# Summary of previous class (08.09)

## THE STANDARD COSMOLOGICAL MODEL

i) matter: $\Omega_{m,0} h^2 \sim 0.14$

ii) Hubble constant: $h \sim 0.67$

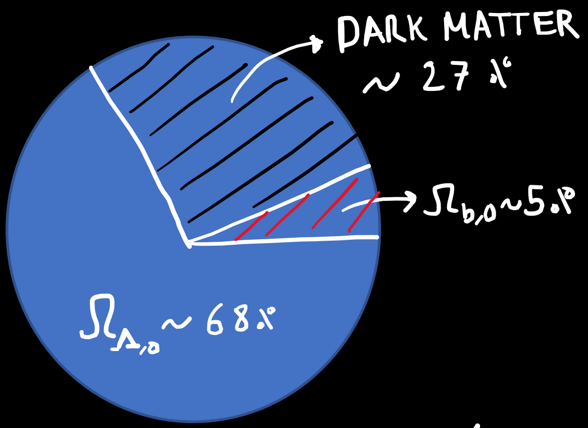
iii) curvature: $\Omega_{K,0} \sim 0$

iv) radiation: $\Omega_{r,0} h^2 \sim 4.2 \times 10^{-5}$

v) baryonic matter: $\Omega_{b,0} h^2 \sim 0.022$

vi) dark matter: $\Omega_{c,0} h^2 \sim 0.12$

vii) cosmological constant: $\Omega_{\Lambda,0} \sim 0.68$




$$\Omega_0 = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0} = 1$$

→ age of the Universe

standard model

$$t_0 = \int_0^\infty \frac{dz}{(1+z)H(z)} \sim 13.8\,\text{Gyr}$$

→ comoving distance: $\chi_e = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_0^z \frac{c\,dz'}{H(z')}$

→ proper distance: $\ell = a_e \chi_e = \frac{a_0 \chi_e}{(1+z)}$

→ angular diameter distance: $d_A = \ell$

→ luminosity distance: $d_L = a_0 \chi_e (1+z) = d_A (1+z)^2$

# Thermodynamics in the expanding Universe: the abundance of thermal relics

From the first law of thermodynamics and some elements of kinetic theory we derived some simple scaling laws valid for the background Universe (density, temperature, … as a function of scale factor)

In this section these results with some elements of statistical mechanics and particle physics to estimate/predict the abundance of certain components in the Universe (known as thermal relics)

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe, the temperatures/densities are sufficiently high ($K_B T > mc^2$) so that particles of mass 'm' can be created and annihilated at similar rates (chemical equilibrium)

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe, the temperatures/densities are sufficiently high ($k_B T > mc^2$) so that particles of mass 'm' can be created and annihilated at similar rates (chemical equilibrium)

e.g. $\underbrace{e^+ + e^-}_{\text{electron-positron annihilation}} \longleftrightarrow \underbrace{\gamma + \gamma}_{\text{photon pair creation}}$ both balanced if $T > \underbrace{511\ \text{KeV}}_{e^-\ \text{rest mass}}$

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe, the temperatures/densities are sufficiently high ($K_B T > mc^2$) so that particles of mass 'm' can be created and annihilated at similar rates (chemical equilibrium)

  e.g. $e^+ + e^- \longleftrightarrow \gamma + \gamma$ both balanced if $T > 511$ KeV

- even if temperatures/densities drop (as the Universe expands and cools down), frequent scattering/interactions between the different particle species mantains thermal (kinetic) equilibrium for some longer time

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe, the temperatures/densities are sufficiently high ($K_B T > mc^2$) so that particles of mass 'm' can be created and annihilated at similar rates (chemical equilibrium)

e.g. $e^+ + e^- \longleftrightarrow \gamma + \gamma$ both balanced if $T > 511$ KeV

- even if temperatures/densities drop (as the Universe expands and cools down), frequent scattering/interactions between the different particle species mantains thermal (kinetic) equilibrium for some longer time

e.g. $\gamma + e^- \longrightarrow \gamma + e^-$ Compton scattering

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe ($K_B T > mc^2$)
  creation/annihilation rates balanced (chemical eq.)
- chemical equilibrium breaks (chemical decoupling)
- scattering processes keep thermal eq.
- thermal/kinetic equilibrium breaks (kinetic decoupling)

Equilibrium Thermodynamics

↓

Out-of-equilibrium Thermodynamics

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe ($k_B T > mc^2$) creation/annihilation rates balanced (chemical eq.)
- chemical equilibrium breaks (chemical decoupling)
- scattering processes keep thermal eq.
- thermal/kinetic equilibrium breaks (kinetic decoupling)

Equilibrium Thermodynamics

↓

Out-of-equilibrium Thermodynamics

→ The transition towards non-equilibrium (decoupling) is roughly given by:

$$\underbrace{\Gamma}_{\substack{\text{interaction rate} \\ \text{(\# of interactions per unit time)}}} \equiv \underbrace{n}_{\text{number density}} \underbrace{\langle \sigma v \rangle}_{\substack{\text{average interaction cross section} \\ \text{times pair relative velocity}}} = \underbrace{H}_{\substack{\text{rate of expansion} \\ \text{of the Universe}}}$$

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe ($k_B T > mc^2$)

  creation/annihilation rates balanced (chemical eq.)
- chemical equilibrium breaks (chemical decoupling)
- scattering processes keep thermal eq.
- thermal/kinetic equilibrium breaks (kinetic decoupling)

} Equilibrium Thermodynamics

↓

Out-of-equilibrium Thermodynamics

→ The transition towards non-equilibrium (decoupling) is roughly given by:

i) $\Gamma \equiv n\langle\sigma v\rangle > H$ (interactions are frequent enough to keep eq.)

ii) $\Gamma \equiv n\langle\sigma v\rangle < H$ (interactions cease to keep eq.)

$\Rightarrow$ particle species decouples from others keeping its comoving density frozen

# Thermodynamics in the expanding Universe: the abundance of thermal relics

→ In a nutshell, this is how this process works:

- in the very early Universe ($k_B T > mc^2$)
  creation/annihilation rates balanced (chemical eq.)
- chemical equilibrium breaks (chemical decoupling)
- scattering processes keep thermal eq.
- thermal/kinetic equilibrium breaks (kinetic decoupling)

} Equilibrium Thermodynamics ↓ Out-of-equilibrium Thermodynamics

→ The transition towards non-equilibrium (decoupling) is roughly given by:

$$\boxed{\Gamma \equiv n\langle \sigma v\rangle = H}$$ → related to abundance of species

→ The physics around decoupling is challenging. It is needed to calculate the final (relic) abundance of stable particles (thermal relics)

# Particles in thermodynamic equilibrium

→ In the very very early Universe al particles where in thermal equilibrium, we then review some fundamentals of statistical mechanics in equilibrium that we need to use

# Particles in thermodynamic equilibrium

Phase space distribution function $\equiv$ phase space density

$dN = f(\vec{x}, \vec{p}, t)\, d^3\vec{x}\, d^3\vec{p} \equiv$ # of particles with positions and momenta within the phase space volume $d^3\vec{x}\, d^3\vec{p}$ centred at $\vec{x}$ and $\vec{p}$

$\rightarrow$ Because of the cosmological principle: $f(\vec{x}, \vec{p}, t) \rightarrow f(p, t)$

$\rightarrow$ Because the system is in thermodynamic eq: $f(p, t) \rightarrow f(p)$

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_B T} \pm 1}\right]}$$

general form of the equilibrium distribution function

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_BT} \pm 1}\right]}$$

→ $g \equiv$ internal degrees of freedom (degeneracy factor)

- molecules : vibration, rotation ...
- elementary particles : spin

(degeneracy in counting particles with the same energy $E(p)$ but different spin)

→ e.g $e^-$ : $g_e = 2$

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_B T} \pm 1}\right]}$$

$\rightarrow$ $g \equiv$ internal degrees of freedom (degeneracy factor)

$\rightarrow$ $\boxed{E(p) = m^2c^4 + p^2c^2}$ energy-momentum relation

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_B T} \pm 1}\right]}$$

→ $g \equiv$ internal degrees of freedom (degeneracy factor)

→ $E(p) = m^2c^4 + p^2c^2$ energy-momentum relation

→ $\boxed{\frac{1}{(2\pi\hbar)^3} \equiv \text{minimum phase space volume}}$ (due to Heisenberg's uncertainty principle, the phase space volume must be measured in units of $h^3 = (2\pi\hbar)^3$)

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_B T} \pm 1}\right]}$$

→ $g \equiv$ internal degrees of freedom
(degeneracy factor)

→ $E(p) = m^2c^4 + p^2c^2$  energy-momentum relation

→ $\frac{1}{(2\pi\hbar)^3} \equiv$ minimum phase space volume

→ $\boxed{\mu \equiv \text{chemical potential}}$

energy absorbed or released during interactions with other particle species that change the particle number

$\boxed{\text{'}\mu\text{' should be conserved in an interaction to have chemical equilibrium}}$

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_B T} \pm 1}\right]}$$

→ $g \equiv$ internal degrees of freedom (degeneracy factor)

→ $E(p) = m^2c^4 + p^2c^2$ energy-momentum relation

→ $\frac{1}{(2\pi\hbar)^3} \equiv$ minimum phase space volume

→ $\boxed{\mu \equiv \text{chemical potential}}$

energy absorbed or released during interactions with other particle species that change the particle number

Bremmsstrahlung radiation

$e^- + p^+ \rightarrow e^- + p^+ + \gamma$

$\mu_e + \mu_p = \mu_e + \mu_p + \mu_\gamma$

$\Rightarrow \boxed{\mu_\gamma = 0}$ → in general for all interactions

e.g.

'$\mu$' should be conserved in an interaction to have chemical equilibrium

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_B T} \pm 1}\right]}$$

$\rightarrow$ $\frac{1}{(2\pi\hbar)^3} \equiv$ minimum phase space volume

Bremmsstrahlung radiation

$e^- + p^+ \rightarrow e^- + p^+ + \gamma$

$\mu_e + \mu_p = \mu_e + \mu_p + \mu_\gamma$

$\Rightarrow \boxed{\mu_\gamma = 0}$ $\rightarrow$ in general for all interactions

needed also to have Planck's law

$\rightarrow$ $g \equiv$ internal degrees of freedom (degeneracy factor)

$\rightarrow$ $E(p) = m^2c^4 + p^2c^2$ energy-momentum relation

$\rightarrow$ $\boxed{\mu \equiv \text{chemical potential}}$

energy absorbed or released during interactions with other particle species that change the particle number

e.g.

'$\mu$' should be conserved in an interaction to have chemical equilibrium

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_BT} \pm 1}\right]}$$

→ $g \equiv$ internal degrees of freedom (degeneracy factor)

→ $E(p) = m^2c^4 + p^2c^2$ energy-momentum relation

→ $\frac{1}{(2\pi\hbar)^3} \equiv$ minimum phase space volume

→ $\mu \equiv$ chemical potential $(\mu_\gamma = 0)$

→ $\pm 1$ in brackets $\begin{cases} +1 & \text{Fermi-Dirac statistics (fermions)} \\ -1 & \text{Bose-Einstein statistics (bosons)} \end{cases}$

Non-relativistic limit

$\boxed{E \sim mc^2 >> k_BT}$

→ Quantum statistics reduces to Classical statistics when: $\exp\left[\frac{(E-\mu)}{kT}\right] >> 1$

# Particles in thermodynamic equilibrium

Phase space distribution function in equilibrium

$$\boxed{f(p) = \frac{g}{(2\pi\hbar)^3}\left[\frac{1}{e^{(E(p)-\mu)/k_B T} \pm 1}\right]}$$

→ $g \equiv$ internal degrees of freedom (degeneracy factor)

→ $E(p) = m^2c^4 + p^2c^2$ energy-momentum relation

→ $\frac{1}{(2\pi\hbar)^3} \equiv$ minimum phase space volume

→ $\mu \equiv$ chemical potential ($\mu_\gamma = 0$)

→ $\pm 1$ in brackets $\begin{cases} +1 & \text{Fermi-Dirac statistics (fermions)} \\ -1 & \text{Bose-Einstein statistics (bosons)} \end{cases}$

→ Non-relativistic limit $mc^2 \gg k_B T$ ⟹ Maxwell Boltzman distribution $f_{MB}(p)$ (if $\mu = 0$)

# Particles in thermodynamic equilibrium

Macroscopic thermodynamic variables from the distribution function

→ number density $\equiv n = \int f(p)\, d^3\vec{p}$ (mass density $\equiv mn$)

→ energy density $\equiv \rho c^2 = \int E(p)\, f(p)\, d^3\vec{p}$

→ Pressure $\equiv P = \underbrace{\frac{1}{3} n \langle |\vec{p}||\vec{v}| \rangle}_{\text{kinetic theory}} = \underbrace{\frac{1}{3} n \left\langle \frac{p^2 c^2}{E} \right\rangle}_{\text{relativistic}} = \int \left( \frac{p^2 c^2}{3E} \right) f(p)\, d^3\vec{p}$

# Particles in thermodynamic equilibrium

Macroscopic thermodynamic variables from the distribution function

$\rightarrow$ number density $\equiv n = \int f(p)\, d^3\vec{p}$ (mass density $\equiv mn$)

$\rightarrow$ energy density $\equiv \rho c^2 = \int E(p)\, f(p)\, d^3\vec{p}$

$\rightarrow$ Pressure $\equiv P = \frac{1}{3} n \langle |\vec{p}||\vec{v}| \rangle = \frac{1}{3} n \left\langle \frac{p^2 c^2}{E} \right\rangle = \int \left( \frac{p^2 c^2}{3E} \right) f(p)\, d^3\vec{p}$

Notes:

i) $d^3\vec{p} = 4\pi p^2 dp$ (isotropic Universe)

ii) change of variable from 'p' to 'E': $E^2 = m^2c^4 + p^2c^2 \Rightarrow 2EdE = 2c^2 p dp$

$\Rightarrow p^2 dp = p \left( \frac{E}{c^2} dE \right) = \left( \frac{E^2 - m^2c^4}{c^2} \right)^{1/2} \left( \frac{EdE}{c^2} \right)$

iii) "natural" units used frequently for this area of cosmology:

$c = \hbar = k_B = 1$

# Particles in thermodynamic equilibrium

iii) "natural" units used frequently for this area of cosmology:

$$c = \hbar = k_B = 1$$

→ They simplify mathematical expressions

→ Without practice, it can be hard to recover the physical dimensions at the end of a calculation

# Particles in thermodynamic equilibrium

iii) "natural" units used frequently for this area of cosmology:

$$c = \hbar = k_B = 1$$

→ For this part of the course:

1) masses: commonly written in eV (electronvolt: Kinetic energy gained by an electron in an electric potential of one Volt $= q_e \times 1$ Volt)

since $E_{rest} = m c^2$

$\Rightarrow m = \frac{E_{rest}}{c^2} \Rightarrow$ units of $m \equiv eV/c^2$

mass-eV: $1\, eV/c^2 = 1.78 \times 10^{-36}$ Kg

If $c = 1$ how do I convert the mass that appears in an equation in eV into Kg?

$1\, KeV = 10^3\, eV$

$m_e = 511$ KeV → multiply by $\frac{1}{c^2} = 1.78 \times 10^{-36}$ Kg/eV

$\Rightarrow m_e \sim 9.1 \times 10^{-31}$ Kg

# Particles in thermodynamic equilibrium

iii) "natural" units used frequently for this area of cosmology:

$$c = \hbar = k_B = 1$$

→ For this part of the course:

2) Temperature: commonly written in eV

since $E_{thermal} = k_B T \Rightarrow T = \frac{E_{Thermal}}{c^2}$ units of $T \equiv eV/k_B$

temp-eV: $1\,eV/k_B = 1.16 \times 10^4\,K$

If $k_B = 1$ how do I convert the temperature that appears in an equation in eV into K?

$1\,MeV = 10^6\,eV$

$T = 30\,MeV \longrightarrow$ multiply by $1/k_B = 1.16 \times 10^4\,K/eV$

$\Rightarrow T \sim 3.48 \times 10^{11}\,K$

# Particles in thermodynamic equilibrium

iii) "natural" units used frequently for this area of cosmology:

$$c = \hbar = k_B = 1$$

→ For this part of the course:

3) time commonly written in eV

→ It comes from $E = h\nu \Rightarrow \nu = \frac{E}{h}$ so frequencies $= \frac{1}{\text{time}}$ can be written as energies if $h = 1$

time-eV : $\hbar/\text{eV} = 6.6 \times 10^{-16}$ s

If $k_B = 1$ how do I convert the time that appears in an equation in eV into s?

$t = 10^{21}\ \text{MeV}^{-1}$ → multiply by $\hbar = 6.6 \times 10^{-16}\ \text{s} \cdot \text{eV}$

$$\Rightarrow t \sim 0.66\ \text{s}$$