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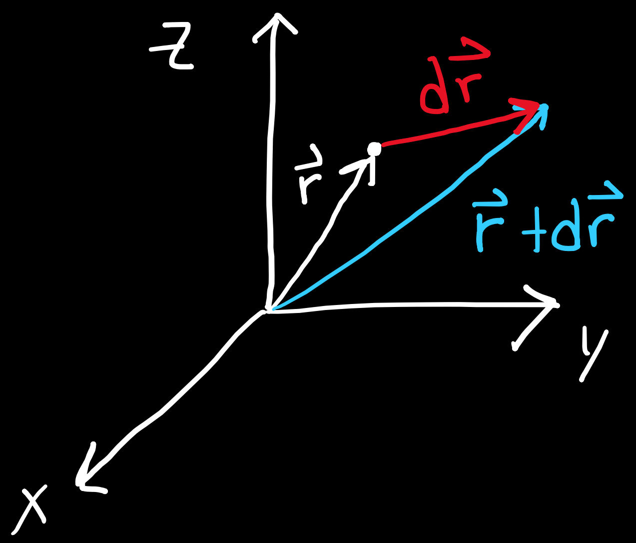

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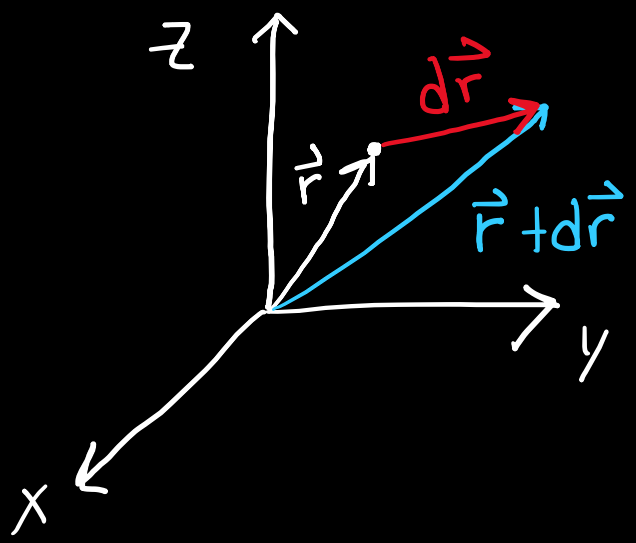

- For instance, the displacement vector $d\vec{r}$ is key to build the kinematics and mechanics in Newtonian Mechanics
- The coordinate values of the displacement vector depend on the coordinate system chosen (cartesian, spherical,..) and the reference frame (whether there is a translation, rotation, Galilean transformation.
- BUT, the magnitude of $|d\vec{r}|$ is an invariant (as we saw in Chapter 1)

i.e. distances are invariant in Newtonian Mechanics

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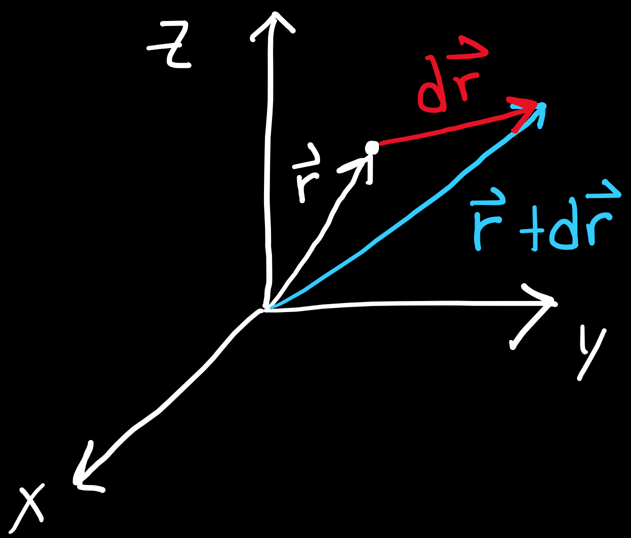

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→ In SR, distances (space intervals) are no longer invariant, instead we have a new invariant quantity: the spacetime interval

$$ds^2 = dr^2 - c^2 dt^2$$

→ "distance" not in Euclidean 3D space but on Minkowski space

4a) Four-vectors in Minkowski space

This implies the need of a new type of vector space in four dimensions that is formulated within Minkowski space and related to the Lorentz transformations

→ Let's introduce key definitions and notation

$\boxed{\text{Four position vector}}$ → space and time are unified in a single vector

$$x^\mu \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \equiv \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

"column vector"

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Four position vector → four displacement vector is simply dx^μ or dx_μ

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"row vector"

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Contravariant- and covariant-like transformations*

* These are, formally, relatively advanced geometrical concepts (part of a course in differential geometry or tensor analysis)
* I will only give an informal introduction. For now, simply take the contravariant and covariant names as two different ways of representing a 4-vector in SR

Contravariant- and covariant-like transformations*

→ The distinction between these two cases is connected to how the <u>components</u> of the 4-vectors transform under a coordinate transformation

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* In the same way as a 3-vector in Euclidean space remains the same vector no matter the coordinate system, a 4-vector in Minkowski space remains the same irrespective of the coordinate system

→ It is the values of the coordinates themselves that are different across different coordinate systems

Contravariant- and covariant-like transformations*

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contravariant: $$V'^{\mu} = \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) V^{\nu}$$

covariant: $$V'_{\mu} = \left(\frac{\partial x_{\mu}}{\partial x'_{\nu}}\right) V_{\nu}$$

Notation: $\mu = 1, 2, 3\ 4$ $\qquad \nu = 1, 2, 3, 4$

$V'^{\mu} \equiv$ contravariant like vector in (S') coordinate system (frame)

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Transformation matrices between coordinate systems

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→ Transformation matrices between coordinate systems

Notation:

→ one index means a vector ; two indices is a tensor (represented by a matrix)

→ Einstein notation for indices: when an index appears twice, it indicates a sum over all the values the index can have (more on this later)

→ position (covariant/contravariant) of main index preserved

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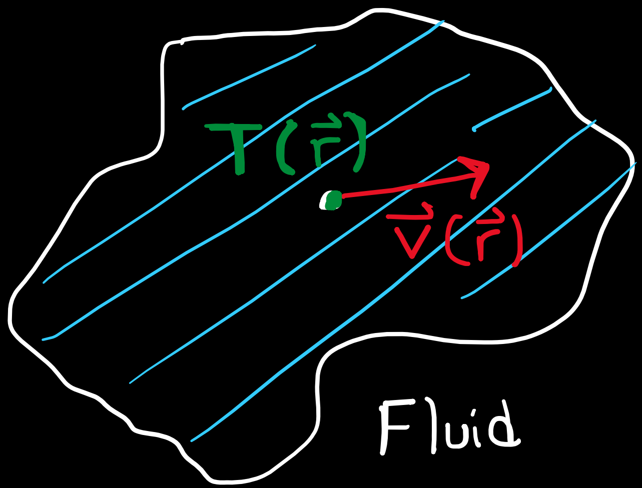

→ Imagine a region in space filled with a fluid (not in thermal equlibrium)

→ Across this region we can define:

1) a temperature field $T(\vec{r})$: a temperature value at each point in space

2) a velocity field $\vec{v}(\vec{r})$: a velocity vector at each point in space

Contravariant- and covariant-like transformations*

1) Covariant case : temperature gradient $\vec{\nabla} T$

→ The temperature gradient is a vector that characterizes how the temperature in the fluid changes across space in different directions

$$\vec{\nabla} T = \frac{\partial T}{\partial x^i} \hat{e}^i = \frac{\partial T}{\partial x} \hat{e}^1 + \frac{\partial T}{\partial y} \hat{e}^2 + \frac{\partial T}{\partial z} \hat{e}^3 \qquad i = 1, 2.3$$

Einstein notation (3D)

$\hat{e}^1 \downarrow \hat{e}_x \qquad \hat{e}^2 \downarrow \hat{e}_y \qquad \hat{e}^3 \downarrow \hat{e}_z \quad \equiv$ unit vectors

→ Einstein notation is a convenient abreviation

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(3D) Einstein notation

$\hat{e}^1 \to \hat{e}_x$, $\hat{e}^2 \to \hat{e}_y$, $\hat{e}^3 \to \hat{e}_z$ $\equiv$ unit vectors

→ Note that we chose cartesian coord., but we could have used other coord., for example spherical

$$\vec{\nabla} T = \frac{\partial T}{\partial x'^i} \hat{e}'^i \quad \text{with} \quad \hat{e}'^1 \equiv \hat{e}_r \quad \hat{e}'^2 \equiv \hat{e}_\theta \quad \hat{e}'^3 \equiv \hat{e}_\varphi$$

$$\text{and} \quad \frac{\partial T}{\partial x'^1} = \frac{\partial T}{\partial r} \equiv T_r \quad \ldots\ldots$$

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$\hat{e}^i \equiv$ Cartesian unit vector basis

$\hat{e}'^i \equiv$ Spherical unit vector basis

Notice that the gradient itself as a 3D vector is the same (magnitude and direction) independent of the coordinate system, but the components of the gradient vector take different values in different coordinate systems

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→ How do the components of the vectors transform across coordinate systems?

e.g. $T_r \equiv \dfrac{\partial T}{\partial r} = \dfrac{\partial T}{\partial x^i} \dfrac{\partial x^i}{\partial r}$ (chain rule + Einstein notation)

$\dfrac{\partial T}{\partial x^i}$: components of $\vec{\nabla} T$ in Cartesian

$\dfrac{\partial x^i}{\partial r}$ → inverse of transformation of coordinates cartesian → spherical

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→ Why the name "covariant"? → $\hat{e}_r = \frac{\partial x^i}{\partial r} \hat{e}_i$ (transformation law for unit vectors)

⇒ The components of the gradient transform in the same way as the unit vector basis. This is where the name co-variant comes from

Contravariant- and covariant-like transformations*

2) <u>Contravariant case</u> : velocity field $\vec{v}(\vec{r})$

$$\vec{v}(\vec{r}) = \frac{dx^i}{dt}\hat{e}_i = \frac{dx}{dt}\underset{\hat{e}_x}{\underset{\downarrow}{\hat{e}_1}} + \frac{dy}{dt}\underset{\hat{e}_y}{\underset{\downarrow}{\hat{e}_2}} + \frac{dz}{dt}\underset{\hat{e}_z}{\underset{\downarrow}{\hat{e}_3}}$$

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(Cartesian Coordinates

In spherical coordinates:

$$\vec{v}(\vec{r}) = \frac{dx'^i}{dt}\hat{e}'_i \qquad \hat{e}'_1 \equiv \hat{e}_r \quad \cdots\cdots$$

→ The transformation for the radial component is:

$$v_r = \frac{dr}{dt} = \frac{\partial r}{\partial x^i}\frac{dx^i}{dt}$$

(Chain rule + Einstein notation)

Transformation of coordinates
Cartesian → Spherical

components of the
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Transformation of coordinates Cartesian → Spherical

Components of the velocity field in Cartesian coordinates

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→ The components of the velocity field transform in the inverse (contra) way as the unit vectors

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Metric: defining four-vector magnitudes

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Minkowski metric

→ a metric is a mathematical object that captures the geometry of spacetime (Minkowski in SR ; it is generalized in GR)

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* The way is written here is valid for the sign convention we chose for the spacetime interval: $ds^2 = dx^2 - c^2dt^2$; for the other sign convention $(-) \leftrightarrow (+)$

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→ For the purposes of this course, it is sufficient to think of the metric as a matrix that defines the magnitude of a four vector

4-vector magnitude

$$\boxed{|V^{\alpha}|^2 = \eta_{\mu\nu} V^{\mu} V^{\nu}}$$

(Einstein notation)

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4-vector magnitude

$$\boxed{|V^\alpha|^2 = \eta_{\mu\nu} V^\mu V^\nu}$$ (Einstein notation)

→ This is analogous to the scalar/dot/inner product in 3D Euclidean geometry

$$|V|^2 = V_x^2 + V_y^2 + V_z^2 \equiv \delta_{ij} V^i V^j \quad ; \; i = 1,2,3$$ (Einstein notation)

with $\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ≡ Kronecker delta → "metric" in Euclidean space

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→ The metric is also used to go from the contravariant to covariant versions of a vector : lowering/rising indices

$$\boxed{X_\mu = \eta_{\mu\nu} X^\nu} \qquad \boxed{X^\mu = \eta^{\mu\nu} X_\nu}$$

metric lowers/rises indices

covariant ↔ contravariant