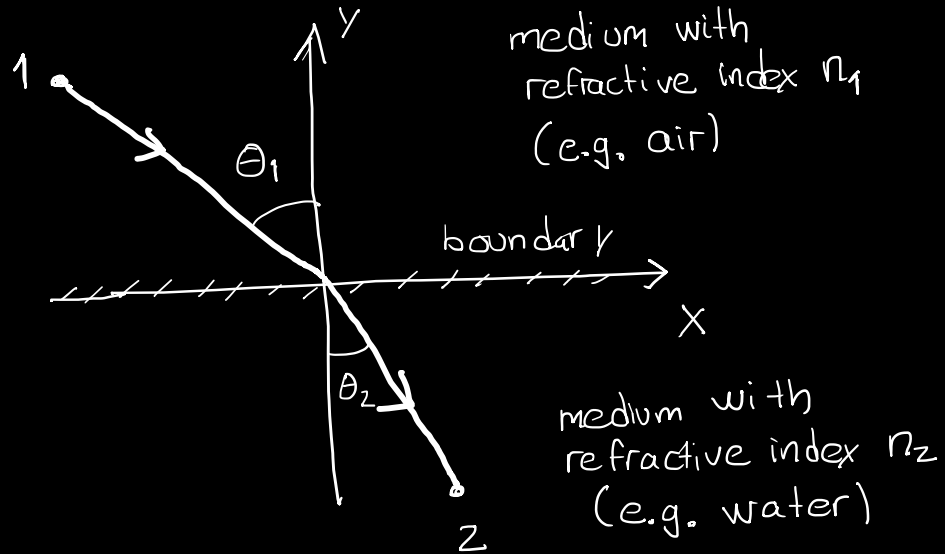


Applications of the Euler-Lagrange equation

2) Fermat's principle



→ Imagine a ray of light traversing two media with different index of refraction (e.g. from air to water)

→ If $n_2 > n_1$, then the speed of light is smaller in the medium with larger index of refraction

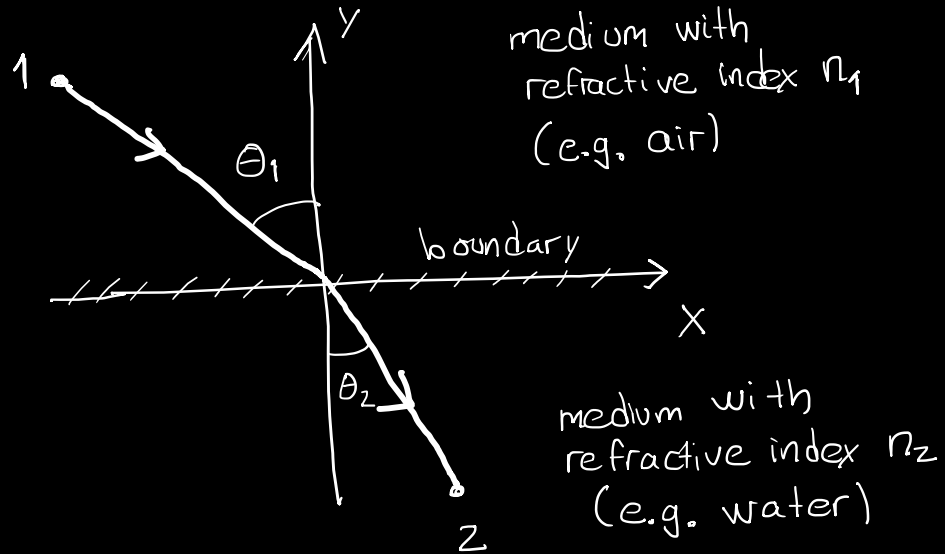
$$c_n = \frac{c}{n} \quad \begin{array}{l} \text{speed of light in vacuum} \\ \Rightarrow c_2 < c_1 \end{array}$$

→ The incidence angle θ_1 and the refraction angle θ_2 are related through:

$$\boxed{n_1 \sin \theta_1 = n_2 \sin \theta_2} \quad \text{Snell's law}$$

Applications of the Euler-Lagrange equation

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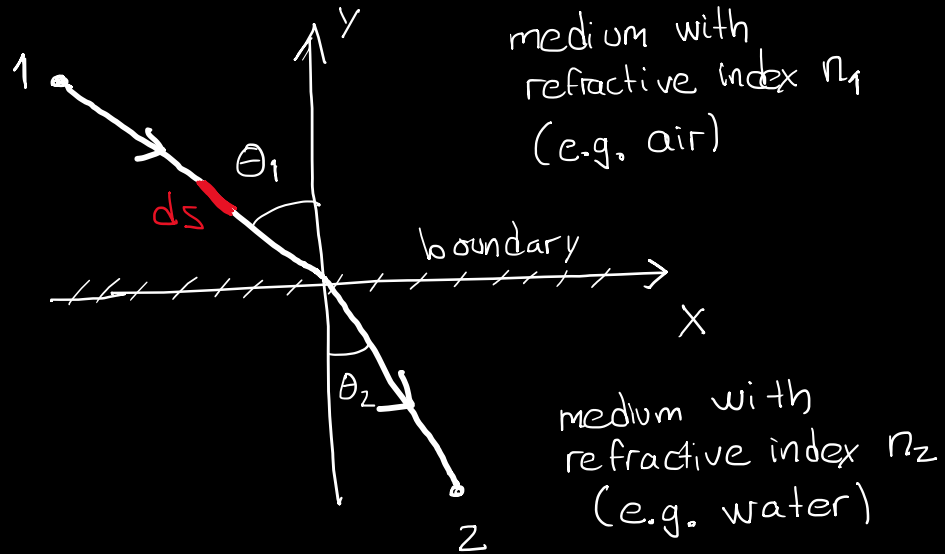
$$\boxed{n_1 \sin \theta_1 = n_2 \sin \theta_2} \quad \text{Snell's law}$$

Goal: derive Snell's law using Fermat's principle:

"The path taken by a ray of light between two points (1 and 2) is a stationary path of the time of passage functional among all possible paths joining 1 and 2"

Applications of the Euler-Lagrange equation

2) Fermat's principle



→ Fermat's principle is in essence a problem for calculus of variations

→ We start by defining the time of passage functional between 1 and 2

$$T = \int_1^2 \frac{ds}{c_n(y)}$$

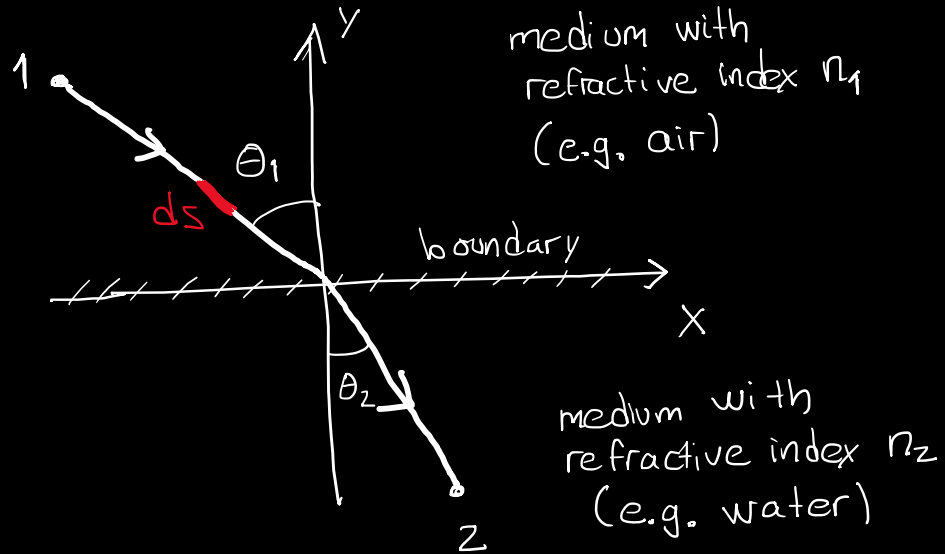
$ds \equiv$ infinitesimal length segment

$c_n(y) \equiv$ speed of light in medium

* Notice: $c_n(y) = \frac{c}{n(y)}$ $\equiv$ function of the position along the vertical axis y in the figure

Applications of the Euler-Lagrange equation

2) Fermat's principle



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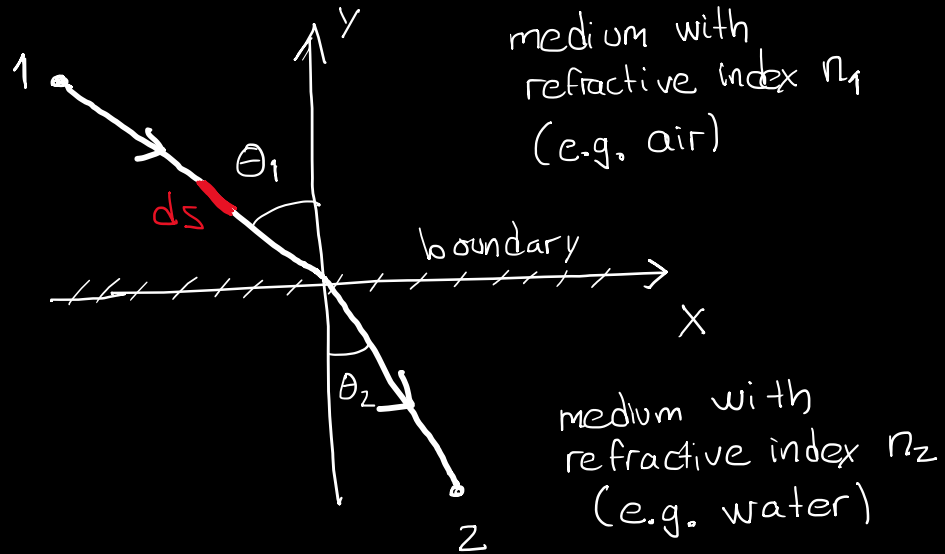
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→ It is convenient to choose the vertical axis as the independent variable since $n \equiv n(y)$

$$\Rightarrow ds = (dx^2 + dy^2)^{1/2} = dy(1 + x'^2)^{1/2} ; \text{ where } x' = \frac{dx}{dy}$$

Applications of the Euler-Lagrange equation

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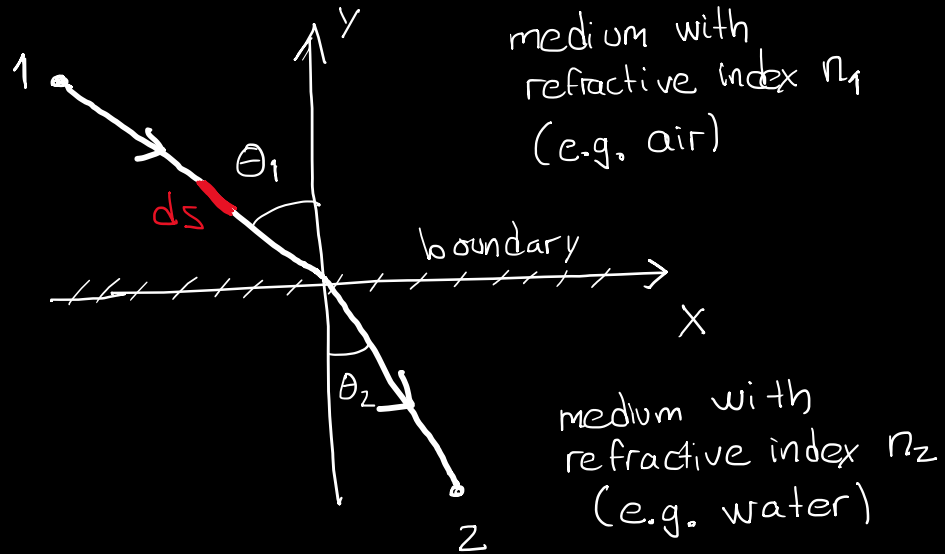
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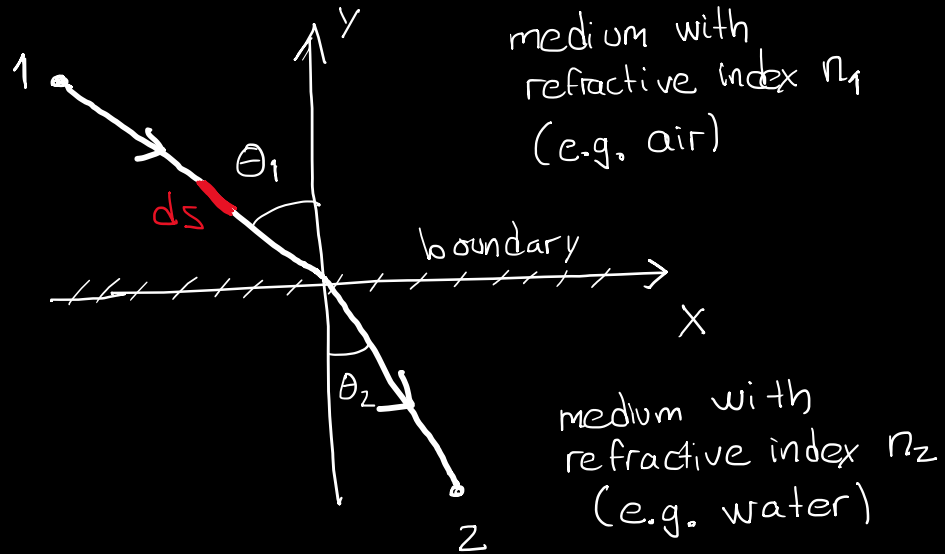
$$\Rightarrow ds = (dx^2 + dy^2)^{1/2} = dy(1 + x'^2)^{1/2} ; \text{ where } x' = \frac{dx}{dy}$$

$$\Rightarrow T[x] = \frac{1}{c} \int_1^2 n(y) (1 + x'^2)^{1/2} dy$$

* notice that the functional T explicitly depends on the dependent variable $x(y)$

Applications of the Euler-Lagrange equation

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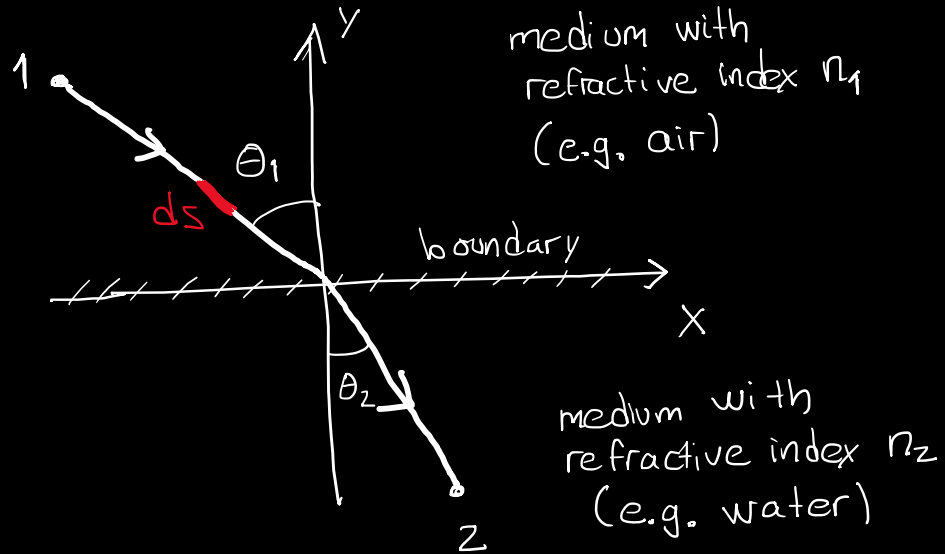
→ Now, we apply the Euler-Lagrange equation properly defining the different quantities:

$$F \equiv F(x'(y); y) \quad x(y) \equiv \text{dependent variable} \quad y \equiv \text{independent variable}$$

$$\Rightarrow \text{Euler-Lagrange eq.} \quad \frac{d}{dy} \left(\frac{\partial F}{\partial x'} \right) - \frac{\partial F}{\partial x} = 0 \quad \text{with} \quad F = \frac{n(y)}{c} (1 + x'^2)^{1/2}$$

Applications of the Euler-Lagrange equation

2) Fermat's principle



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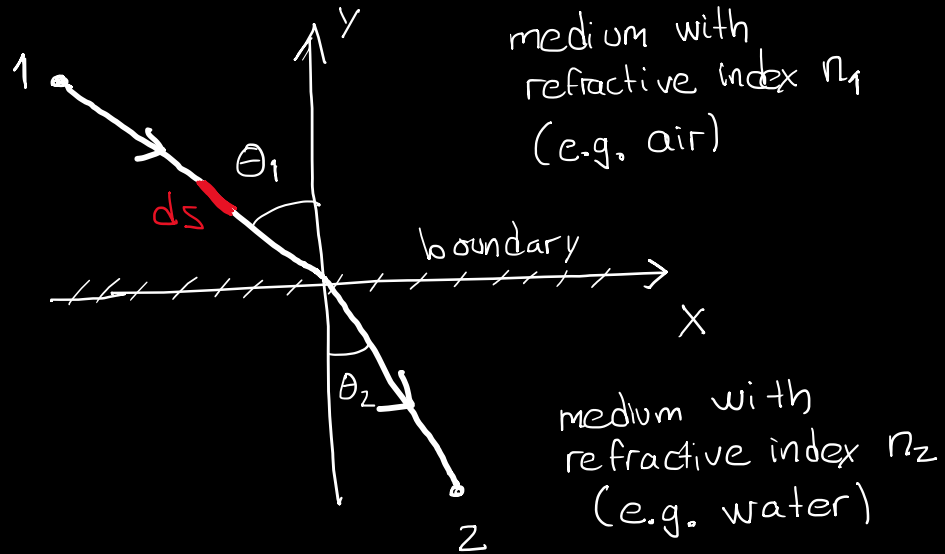
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$$\text{Now: } \frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial x'} = \frac{n(y) x'}{(1 + x'^2)^{1/2}} \quad \therefore \frac{d}{dy} \left[\frac{n(y) x'}{(1 + x'^2)^{1/2}} \right] = 0$$

Applications of the Euler-Lagrange equation

2) Fermat's principle



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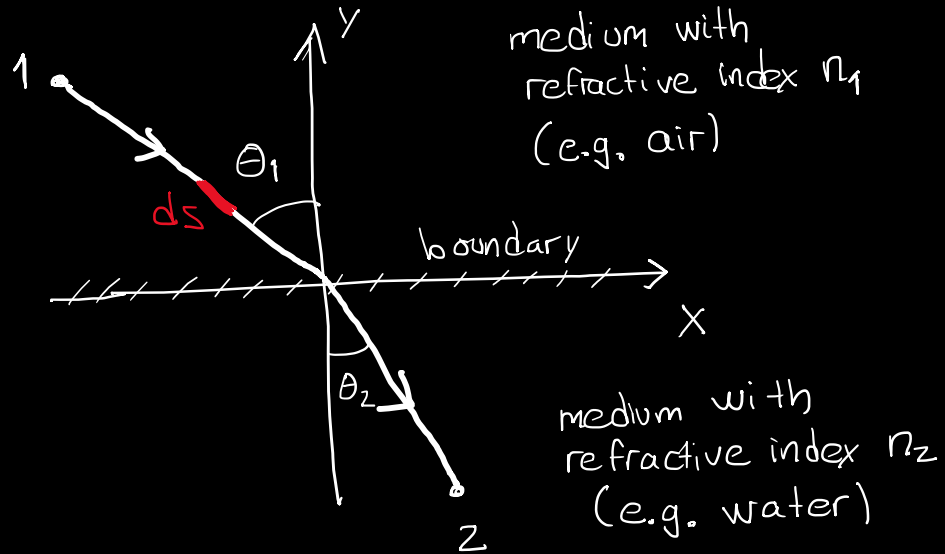
→ But notice from the figure:

$$\frac{x'}{(1 + x'^2)^{1/2}} = \frac{dx}{(dy^2 + dx^2)^{1/2}} = \sin \Theta(y)$$

where $\Theta \equiv \Theta(y) \rightarrow$ "the angle that the path makes with the vertical axis depends on the position along this axis (different media)"

Applications of the Euler-Lagrange equation

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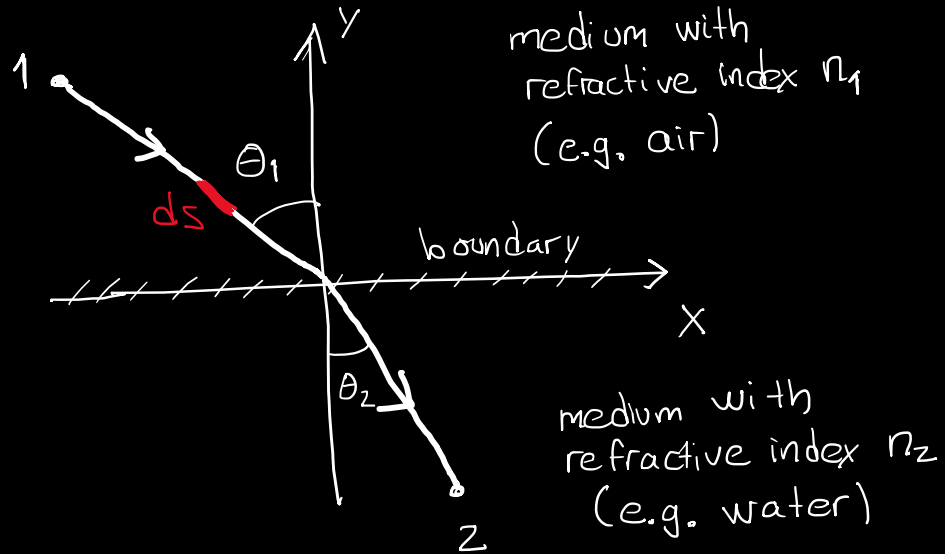
$$\therefore \frac{d}{dy} \left[\frac{n(y) x'}{(1+x'^2)^{1/2}} \right] = 0$$

$$\Rightarrow \frac{d}{dy} [n(y) \sin \Theta(y)] = 0$$

$$\therefore n(y) \sin \Theta(y) \equiv \text{constant}$$

Applications of the Euler-Lagrange equation

2) Fermat's principle



The stationary path $y(x)$ for the time of passage between 1 and 2 has a constant/integral of motion

→ Fermat's principle is in essence a problem for calculus of variations

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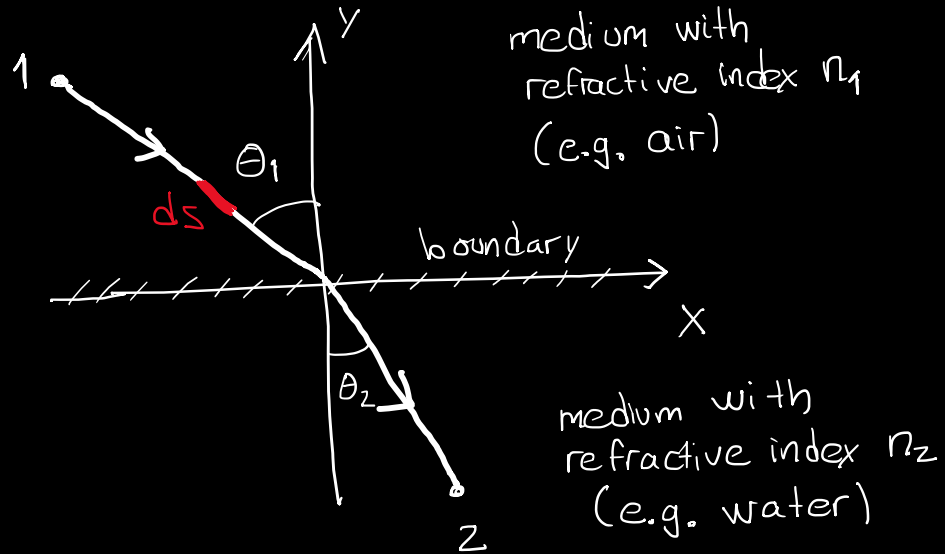
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$$\Rightarrow \therefore n(y) \sin \theta(y) \equiv \text{constant}$$

Applications of the Euler-Lagrange equation

2) Fermat's principle



→ For the case of two media shown in the figure, this leads to:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \text{Snell's law}$$

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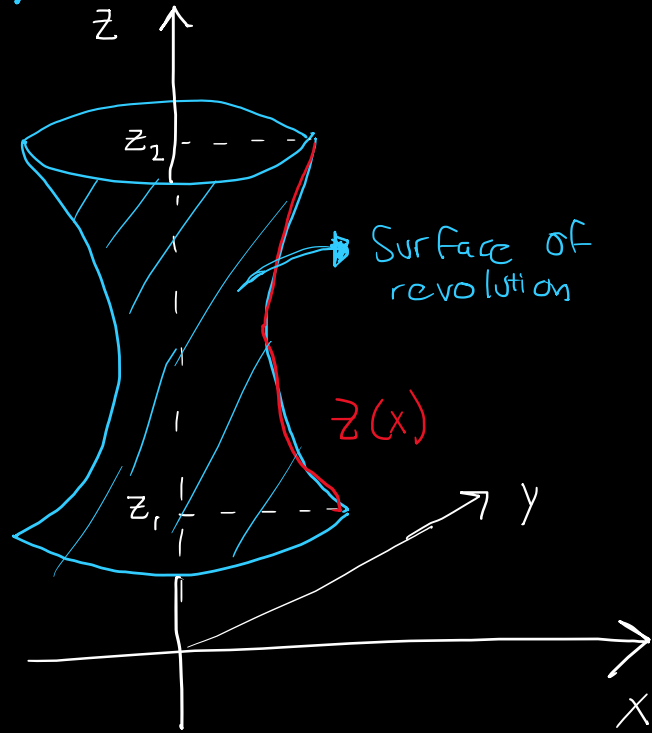
$$n(y) \sin \theta(y) \equiv \text{constant}$$

* This derivation give us an intuitive interpretation of why bends in this way:

If the speed of light is greater in n_1 (e.g. air) than in n_2 (e.g. water) then, in order to minimize the time of passage between 1 and 2, it needs to extend its path in n_1 (e.g. air) and reduce it in n_2 (e.g. water)

Applications of the Euler-Lagrange equation

ii) Minimal surfaces of revolution



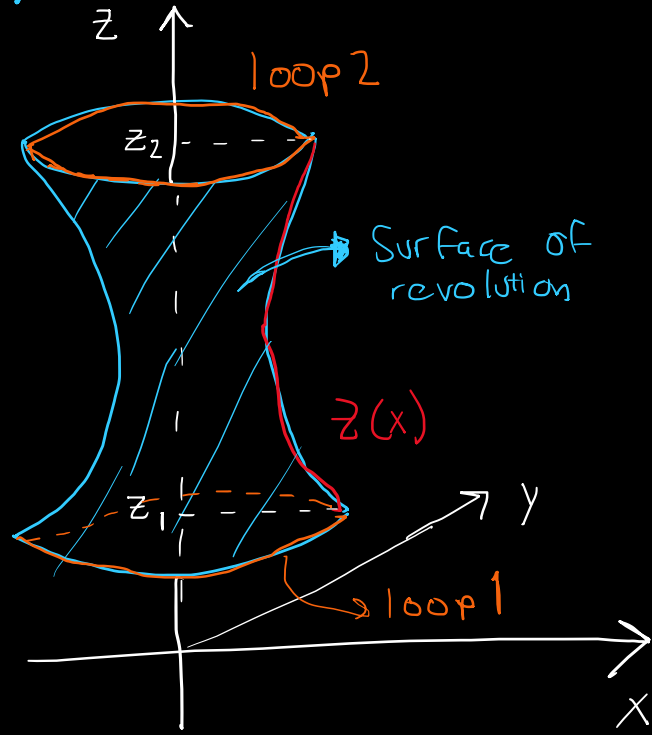
→ The problem consists on finding the surface of revolution generated by revolving a curve $z(x)$ around the z -axis that minimizes the area of the surface

→ Physically, this problem is connected to "soap films" generated for example by dipping a loop of wire into a soap solution and blowing air into it (nature does this by minimizing the surface area of the film)

* In the case of this problem, we can think of a soap bubble/film that stretches between two loops of wire

Applications of the Euler-Lagrange equation

ii) Minimal surfaces of revolution



→ The problem consists on finding the surface of revolution generated by revolving a curve $z(x)$ around the z -axis that minimizes the area of the surface

→ Following the geometry of the figure, we notice:

i) The two loops (circles) are a boundary (top and bottom) of the surface. Let's say these loops are located at:

$$\text{loop 1 : } z = z_1$$

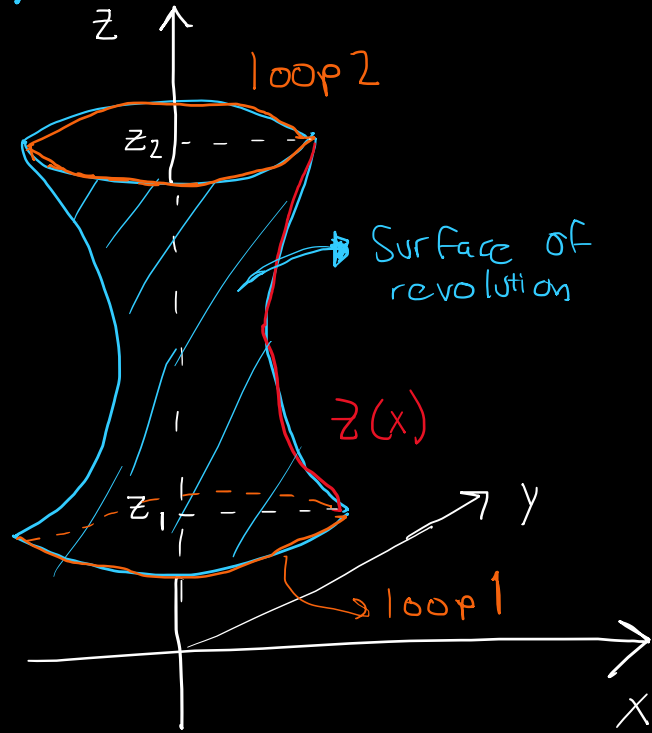
$$\text{loop 2 : } z = z_2$$

ii) Since the z -axis is the axis of symmetry of the surface, it is convenient to work with cylindrical coordinates:

$$(r, \theta, z) \quad \text{where } (r, \theta) \text{ are the polar coordinates}$$

Applications of the Euler-Lagrange equation

ii) Minimal surfaces of revolution



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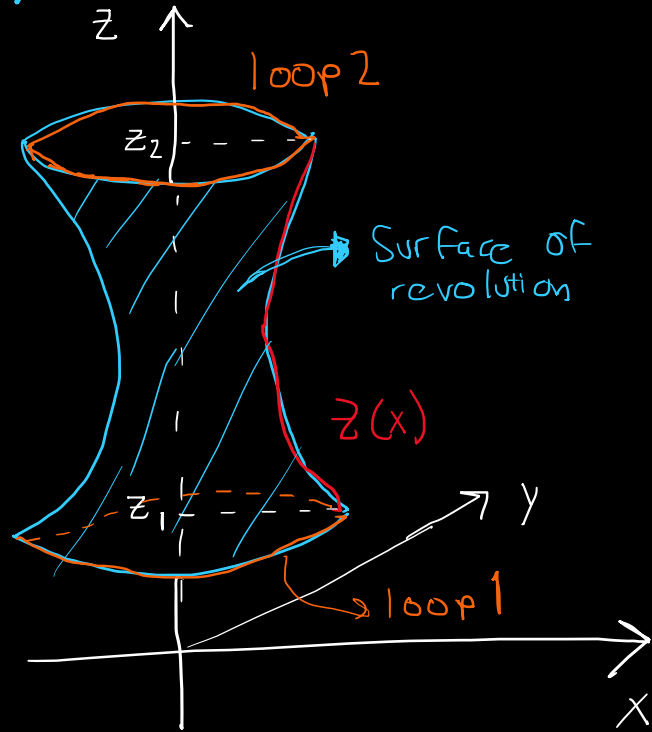
$$(\rho, \theta, z) \quad \text{where } (\rho, \theta) \text{ are the polar coordinates}$$

Note: the problem helps illustrate both that choosing appropriate coordinate system, AND appropriate independent variable to apply variational calculus is quite useful to simplify the solution. But the answer should not depend on the choice!

(to show this, we will consider two choices of ind. variables: ρ and z)

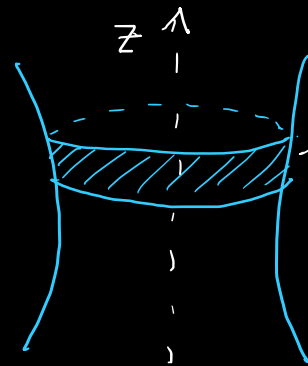
Applications of the Euler-Lagrange equation

iii) Minimal surfaces of revolution

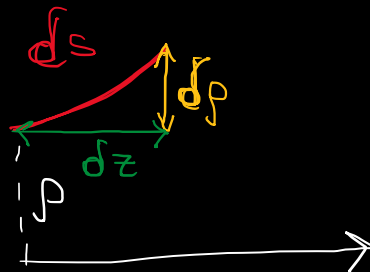
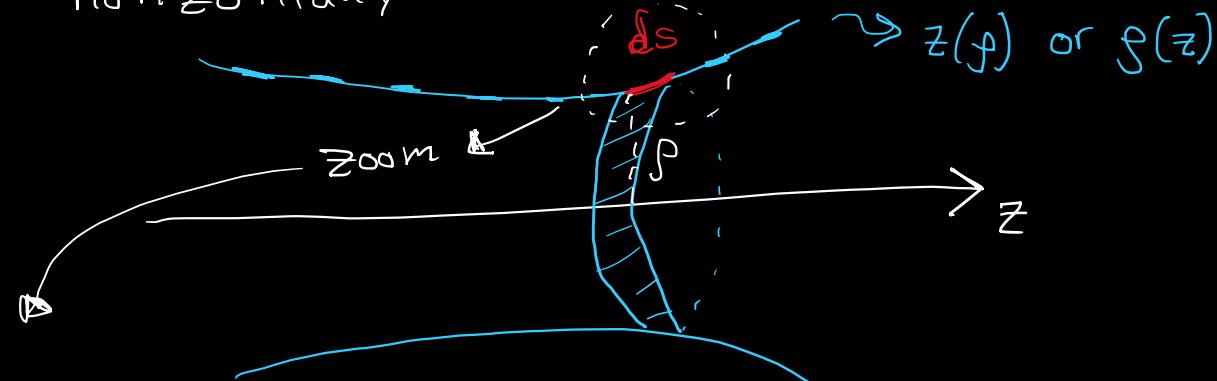


→ The problem consists on finding the surface of revolution generated by revolving a curve $z(x)$ around the z -axis that minimizes the area of the surface

→ We start by deriving the functional that represents the surface of revolution



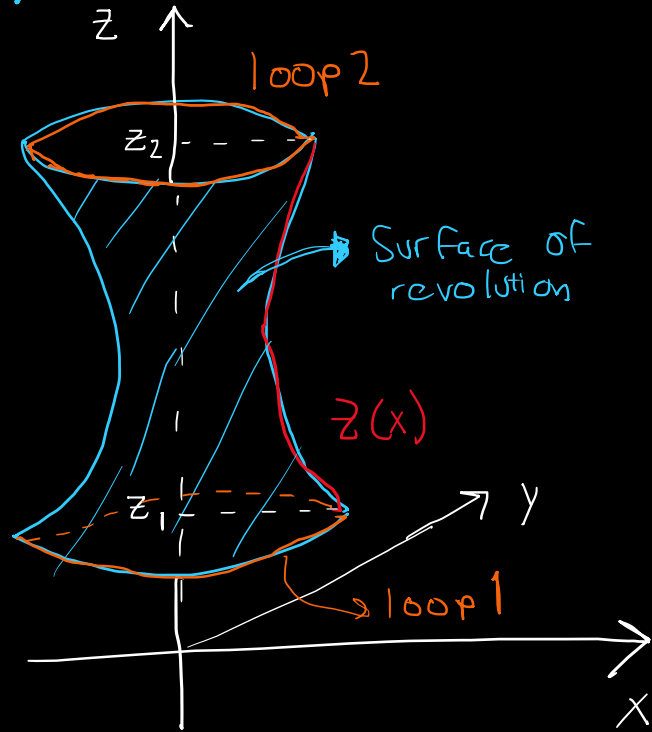
consider a small ring in the surface of revolution and switch the perspective horizontally



→ The thickness of the ring ds is the arc length of the curve $z(p)$ or $p(z)$ depending on choice of ind. variable

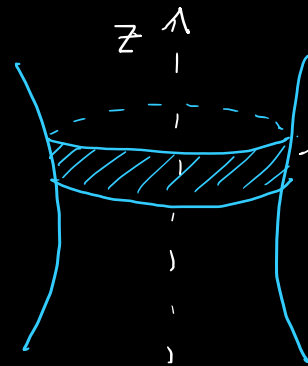
Applications of the Euler-Lagrange equation

ii) Minimal surfaces of revolution

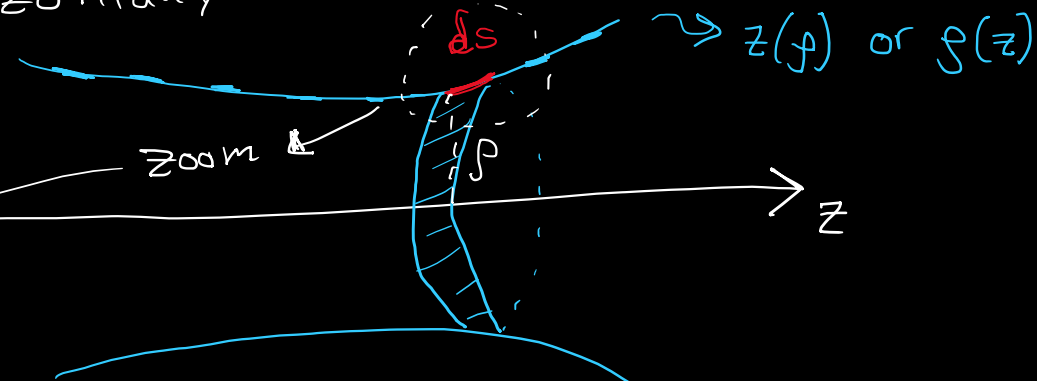


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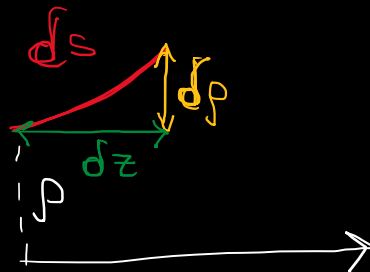
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→ We then have:

$$\delta s^2 \sim \delta p^2 + \delta z^2, \text{ then}$$

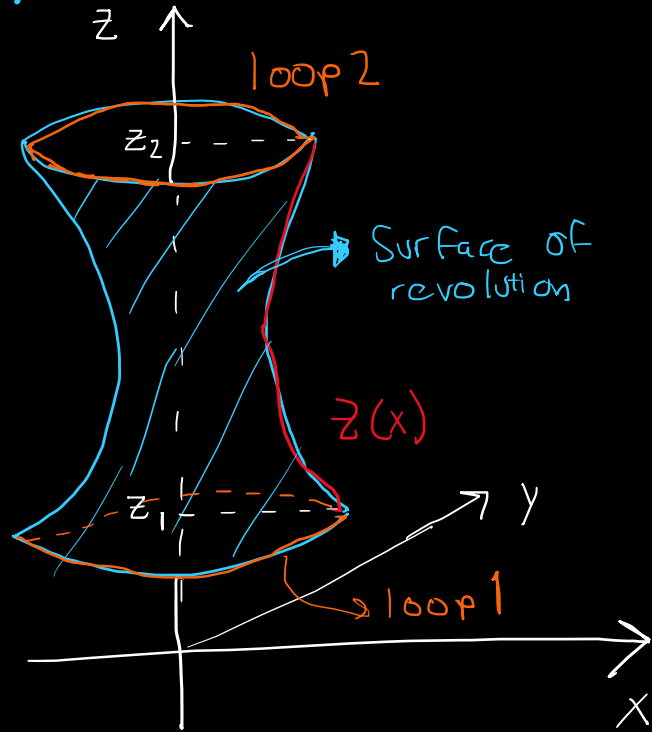
$$ds = \begin{cases} dp (1 + z'(p)^2)^{1/2} & \text{if } p \text{ ind. var.} \\ dz (p'(z) + 1)^{1/2} & \text{if } z \text{ ind. var.} \end{cases}$$



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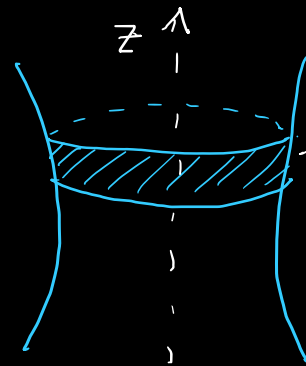
Applications of the Euler-Lagrange equation

ii) Minimal surfaces of revolution

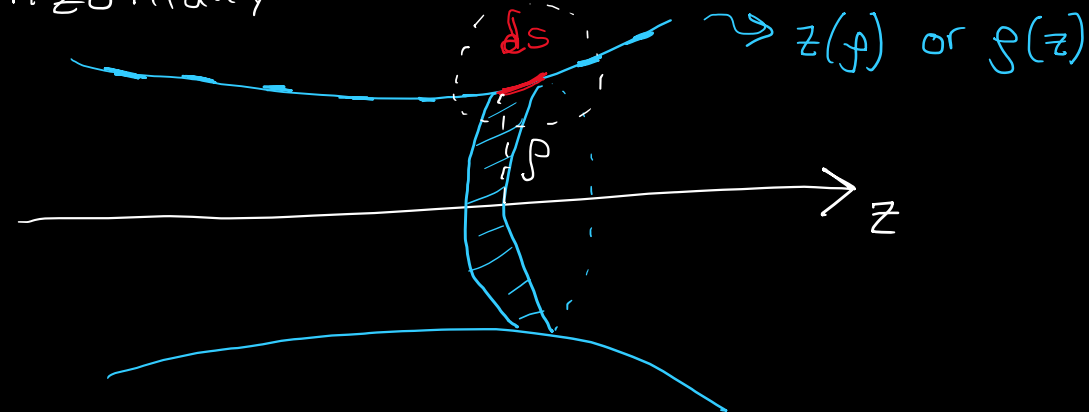


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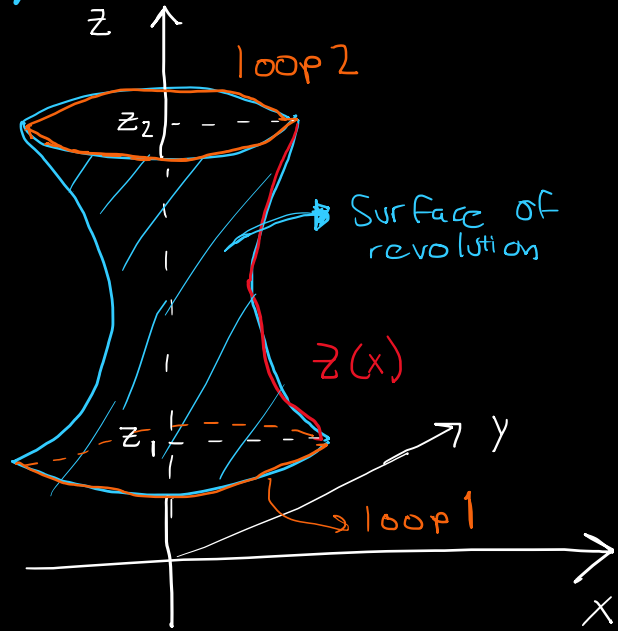
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⇒ area of infinitesimal ring is: $dA = 2\pi p ds$

∴ $A = 2\pi \int_1^2 p ds$ Functional of surface of revolution

Applications of the Euler-Lagrange equation

iii) Minimal surfaces of revolution



→ The problem consists on finding the surface of revolution generated by revolving a curve $z(x)$ around the z -axis that minimizes the area of the surface

→ We start by deriving the functional that represents the surface of revolution

$$A = 2\pi \int_1^2 \rho \, ds$$

$$ds = \begin{cases} d\rho (1 + z'(\rho)^2)^{1/2} & \text{if } \rho \text{ ind. var.} \\ dz (\rho'(z) + 1)^{1/2} & \text{if } z \text{ ind. var.} \end{cases}$$

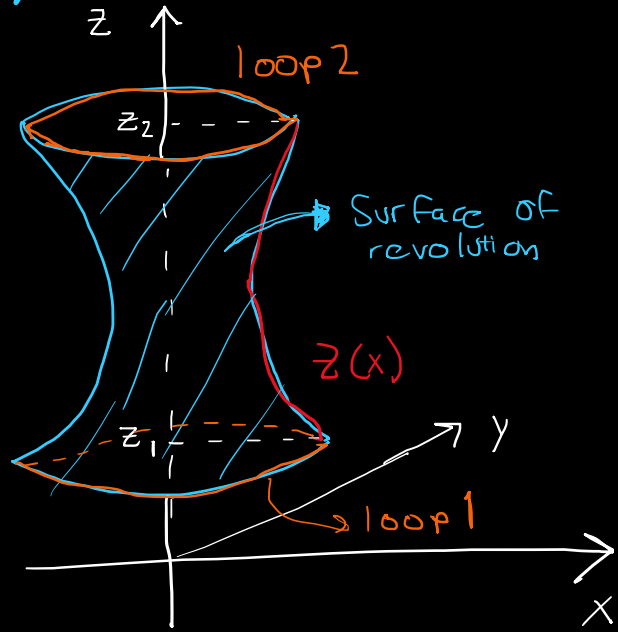
Case 1: choosing z as the independent variable

$$\Rightarrow A[\rho] = 2\pi \int_1^2 \rho (1 + \rho'(z)^2)^{1/2} dz \quad F \equiv \rho (1 + \rho'^2)^{1/2}$$

$$\text{Euler-Lagrange equation: } \frac{d}{dz} \left(\frac{\partial F}{\partial \rho'} \right) - \frac{\partial F}{\partial \rho} = 0$$

Applications of the Euler-Lagrange equation

iii) Minimal surfaces of revolution



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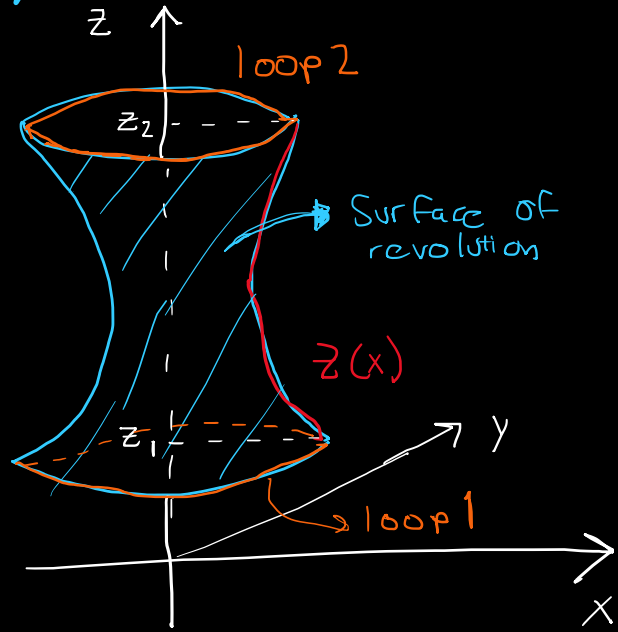
Euler-Lagrange equation: $\frac{d}{dz} \left(\frac{\partial F}{\partial \rho'} \right) - \frac{\partial F}{\partial \rho} = 0$

$$\frac{\partial F}{\partial \rho} = (1 + \rho'^2)^{1/2} \quad \frac{\partial F}{\partial \rho'} = \frac{\rho \rho'}{(1 + \rho'^2)^{1/2}} \Rightarrow \left[\frac{d}{dz} \left(\frac{\rho \rho'}{(1 + \rho'^2)^{1/2}} \right) - (1 + \rho'^2)^{1/2} \right] = 0$$

Not easy to solve!

Applications of the Euler-Lagrange equation

iii) Minimal surfaces of revolution



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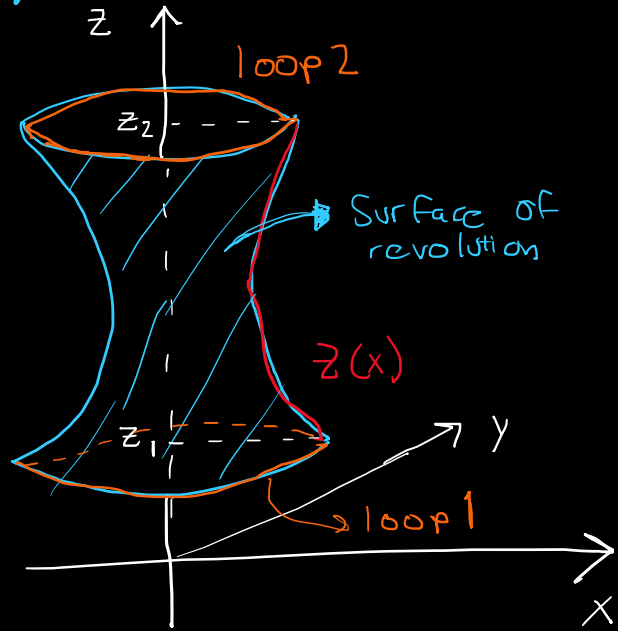
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Euler-Lagrange equation: $\frac{d}{d\rho} \left(\frac{\partial F}{\partial z'} \right) - \frac{\partial F}{\partial z} = 0$

Applications of the Euler-Lagrange equation

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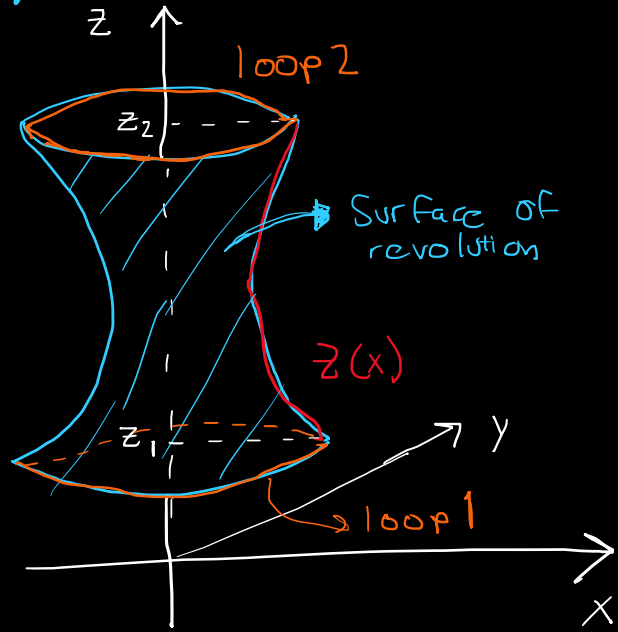
$$\frac{\partial F}{\partial z} = 0 \quad \frac{\partial F}{\partial z'} = \frac{\rho z'}{(1 + z'^2)^{1/2}} \Rightarrow \frac{d}{d\rho} \left(\frac{\rho z'}{(1 + z'^2)^{1/2}} \right) = 0 \Rightarrow$$

can be integrated directly

$$\frac{\rho z'}{(1 + z'^2)^{1/2}} = a \equiv \text{constant}$$

Applications of the Euler-Lagrange equation

iii) Minimal surfaces of revolution



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Case 1: choosing ρ as the independent variable

$$\frac{\rho z'}{(1 + z'^2)^{1/2}} = a \equiv \text{constant}$$

$$\Rightarrow \rho^2 z'^2 = a^2 (1 + z'^2) \Rightarrow z'^2 (\rho^2 - a^2) = a^2$$

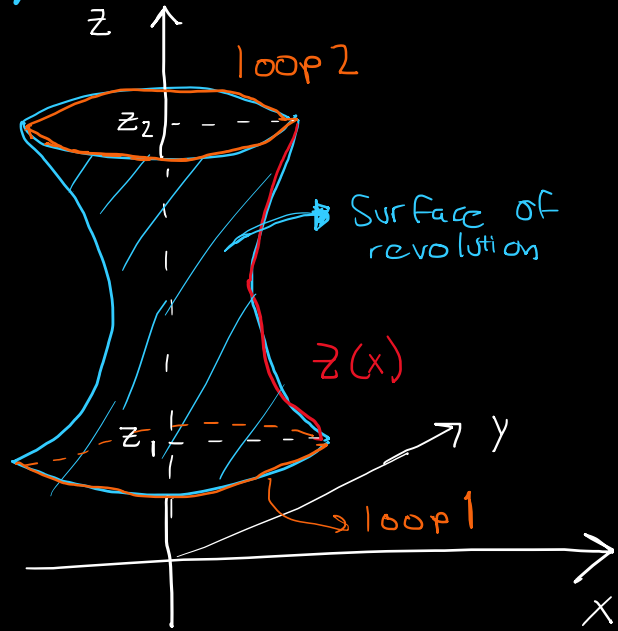
$$\Rightarrow \frac{dz}{d\rho} = \frac{a}{(\rho^2 - a^2)^{1/2}} \Rightarrow \int dz = a \int \frac{d\alpha}{(\alpha^2 - 1)^{1/2}} \quad \alpha = \rho/a$$

→ can be integrated (e.g. integral tables, hyperbolic functions): $z(\rho) = a \cosh^{-1}(\rho/a) + b$

→ hyperbolic cosine
→ constant

Applications of the Euler-Lagrange equation

iii) Minimal surfaces of revolution



→ The problem consists on finding the surface of revolution generated by revolving a curve $z(x)$ around the z -axis that minimizes the area of the surface

→ We start by deriving the functional that represents the surface of revolution

$$A = 2\pi \int_1^2 \rho \, ds$$

$$ds = \begin{cases} d\rho (1 + z'(\rho)^2)^{1/2} & \text{if } \rho \text{ ind. var.} \\ dz (\rho'(z) + 1)^{1/2} & \text{if } z \text{ ind. var.} \end{cases}$$

Case 1: choosing ρ as the independent variable

$$z(\rho) = a \cosh^{-1}(\rho/a) + b$$

$$\frac{\rho z'}{(1 + z'^2)^{1/2}} = a \equiv \text{constant}$$

or $\rho(z) = a \cosh\left(\frac{z-b}{a}\right) \rightarrow$ eq. of a "catenary" (shape that a flexible cable takes under gravity)

